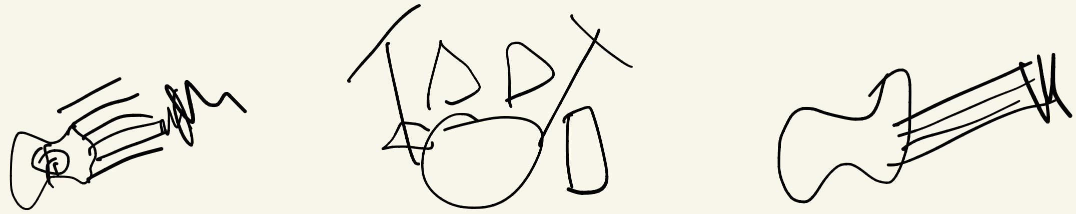
Pb. 1

(a)



$$10 \cdot 9 \cdot 8$$

(b)

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{2}$$

(c)

$$P(X=Y) = \sum_{i=1}^{100} P(X=Y \mid Y=i)\, P(Y=i)$$

$$= \sum_{i=1}^{100} \overbrace{P(X=i \mid \cancel{Y=i})}^{\frac{1}{100}} \underbrace{P(Y=i)}_{\frac{1}{100}} = \cancel{100} \cdot \frac{1}{\cancel{100}} \frac{1}{100} = \frac{1}{100}$$

(a) A = event picked coin is unfair

B = event of 10 heads

Quest. $P(A|B)$

$$= \frac{P(B|A)\,P(A)}{P(B)}$$

$$= \frac{\overset{1}{P(B|A)}\;\overset{\frac{1}{1000}}{P(A)}}{\underbrace{P(B|A)}_{1}\underbrace{P(A)}_{\frac{1}{1000}} + \underbrace{P(B|A^c)}_{\left(\frac{1}{2}\right)^{10}}\underbrace{P(A^c)}_{\frac{999}{1000}}}$$

$1000 \cdot 2^{10}$

$$= \frac{\cancel{2^{10}}\;1000}{2^{10} + 999} \approx 1/2$$

Pb. 2

(a) $$\begin{pmatrix} W \\ X_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\mathrm{Cov}(W, X_2) = \mathrm{Cov}(\sqrt{2} X_1 - X_2, X_2)$$

$$= \sqrt{2}\, \overbrace{\mathrm{Cov}(X_1, X_2)}^{\frac{1}{\sqrt{2}}} - \underbrace{\mathrm{Var}(X_2)}_{1} = 0$$

$\rightarrow$ indep!

(b) $\begin{pmatrix} X_+ \\ X_- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$\begin{cases} X_1, X_2 \overset{iid}{\sim} N(0,1) \\ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \end{cases}$$

I. X_+, X_- is multivariate N

II. $Cov(X_+, X_-) = Cov(X_1 + X_2, X_1 - X_2)$

$= Var(X_1) - Var(X_2)$

$= 0 \rightarrow$ indep

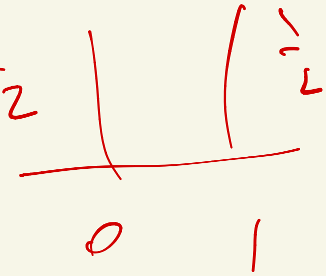
b(i) : not always dep.

counter - ex :

b(ii) : not always indep

counter - ex : iid Bernoulli

$X_1, X_2 \overset{iid}{\sim} \mathrm{Ber}\left(\frac{1}{2}\right)$



if $X_+ = 2$

$\Rightarrow X_1 = X_2 = 1$

$\Rightarrow X_- = 0$

$\Rightarrow$ dependent

(c) $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N}_3\left(\begin{pmatrix} \vdots \end{pmatrix}, \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right)$

$\boxed{\text{NO}}$ $\quad f_{X_1, X_2, X_3} = f_{X_1} f_{X_2} f_{X_3}$

$\Rightarrow$ independence

(d) $X_1 \perp\!\!\!\perp X_2$

$X_1 \perp\!\!\!\perp X_3$

$X_2 \perp\!\!\!\perp X_3$

but (X_1, X_2, X_3) not mutually indep ?

" given X_1, X_2 we can infer X_3 differently than its marginal "

$$\boxed{\begin{aligned} &X_1, X_2 \overset{iid}{\sim} \mathrm{Ber}(\tfrac{1}{2}) \\ &X_3 = X_1 \oplus X_2 \end{aligned}} \sim \mathrm{Ber}(\tfrac{1}{2})$$

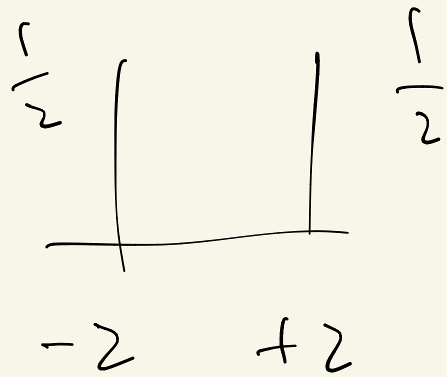
yes!

$\boxed{X_1, X_3 \text{ are indep !}}$

$$P(\cancel{X_1} \oplus X_2 = y \mid X_1 = x) = P(X_1 \oplus X_2 = y)$$

$$P(X_2 = y \oplus x) = \frac{1}{2}$$

Pb. 3



(a) show: $P\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{1}{100}\right)$

$$\le P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right|^2 > \left(\frac{1}{100}\right)^2\right)$$

$$\le \frac{E\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right|^2}{\left(\frac{1}{100}\right)^2}$$

$$= \frac{\frac{1}{n} Var(X_1)}{\left(\frac{1}{100}\right)^2}$$

$$Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} n \, Var(X_1)$$

(b) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i = \tilde{S}_n$

$$M_{\tilde{S}_n}(t) = E e^{t\tilde{S}_n}$$

$$= E e^{\frac{t}{\sqrt{n}} \sum_{i=1}^{n} X_i}$$

$$= \prod_{i=1}^{n} E e^{\frac{tX_i}{\sqrt{n}}}$$

$$= \Big(\underbrace{E e^{\frac{tX_1}{\sqrt{n}}}}\Big)^n$$

$$= \left(\frac{e^{2t/\sqrt{n}} + e^{-\frac{2t}{\sqrt{n}}}}{2}\right)^n$$

$$= \frac{e^{2t\sqrt{n}}\left(1+e^{-\frac{4t}{\sqrt{n}}}\right)^n}{2}$$

(c) $M_{\tilde{S}_n} \to e^{2t^2}$

$$N(0, 4)$$

therefore $\tilde{S}_n \xrightarrow{D} N(0, 4)$

Pb 4) $X_1, \dots, X_n \overset{iid}{\sim} Poi(\theta)$

(a) $EX_1 = \theta$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \theta$$

$$\hat{\theta}_{Mom} = \bar{X}$$

(b) $L(\theta) = \prod_{i=1}^{n} P(X_i = x_i)$

log   log

$$= \prod_{i=1}^{n} \frac{e^{-\theta}\,\theta^{x_i}}{x_i!}$$

log

$$= \sum_{i=1}^{n} \left(x_i \log\theta - \theta - \log x_i! \right)$$

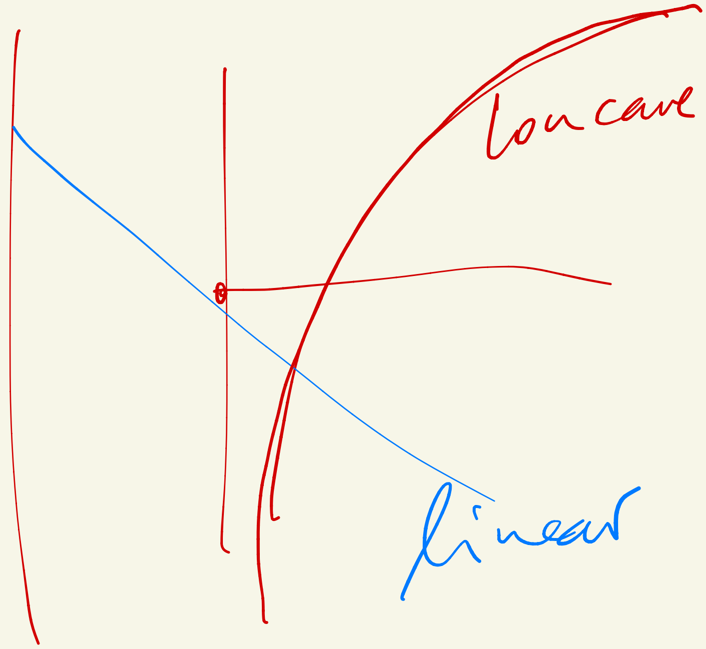
concave

$$= \log\theta \left(\sum_{i=1}^{n} x_i \right) - n\theta + c$$

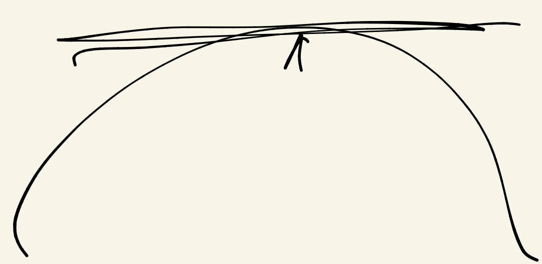
maximize ↑ over θ

$$\boxed{\frac{\partial}{\partial\theta}(\;) = 0}$$

$$\boxed{\frac{\partial^2}{\partial\theta^2}(\;) < 0}$$



$$\left(\sum x_i\right) \frac{1}{\hat\theta_{MLE}} = n$$



$$\boxed{\hat\theta_{MLE} = \bar{X}}$$

$$\frac{\partial^2}{\partial\theta^2} : -\sum_{i=1}^{n} \frac{x_i}{\theta^2} < 0$$

(c) $b(\theta) = E(\hat{\theta}) - \theta$

$= E\left(\frac{1}{n}\sum X_i\right) - \theta$

$= \frac{1}{\cancel{n}} \cancel{n} E X_1 - \theta$

$= \theta - \theta = 0$

$Var(\hat{\theta}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \frac{1}{n^2} n \, Var(X_1)$

$= \frac{1}{n}\theta$

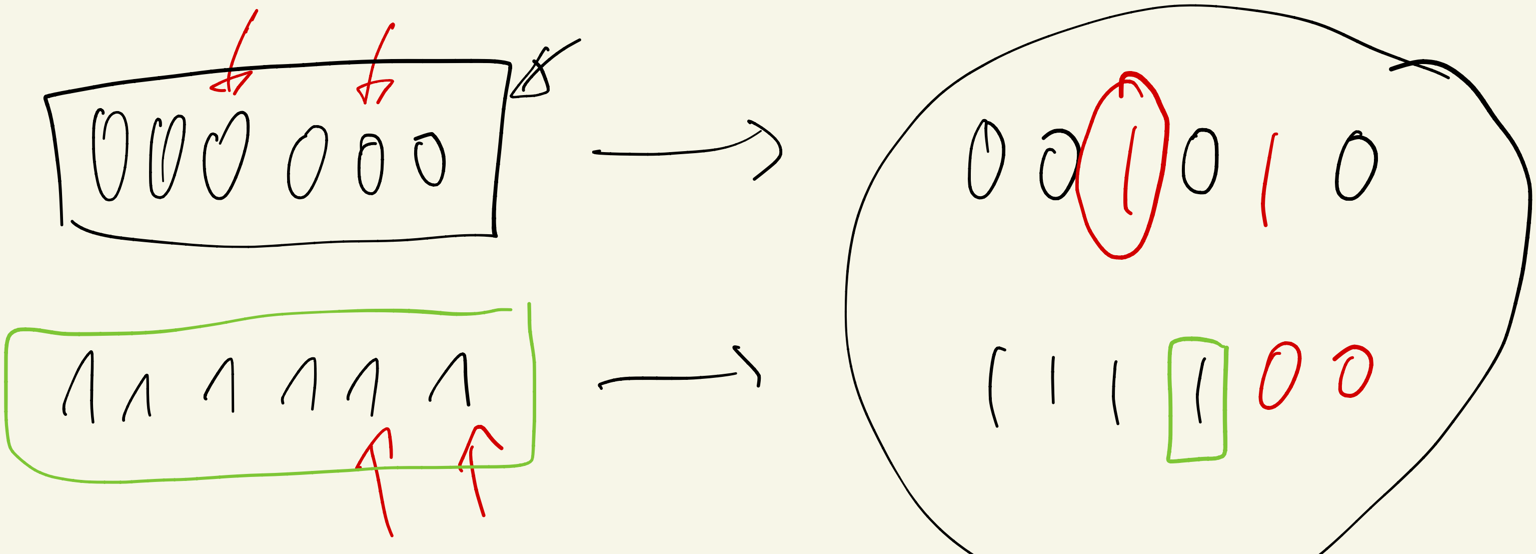
$MSE = var(\hat{\theta}) + b(\theta)^2$

$$E(\hat{\theta}-\theta)^2 = \underline{\operatorname{Var}(\hat{\theta})} + \underline{\operatorname{bias}(\hat{\theta})^2}$$

$$+ E\hat{\theta} - E\hat{\theta}$$

$$= \frac{\theta}{n}$$

Pb(5)



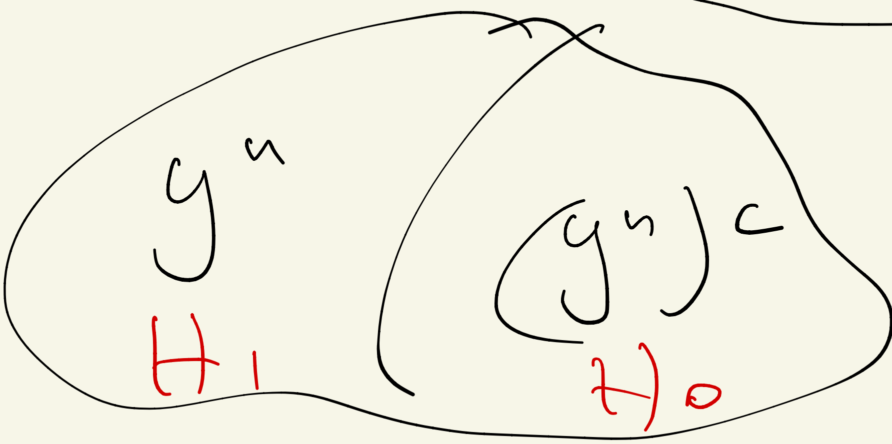
$$Y^n \sim \begin{cases} H_0 : Y^n \overset{iid}{\sim} \mathrm{Ber}(p) \\ H_1 : Y^n \overset{iid}{\sim} \mathrm{Ber}(1-p) \end{cases}$$

$$p \in \left(0, \tfrac{1}{2}\right)$$

False positive :

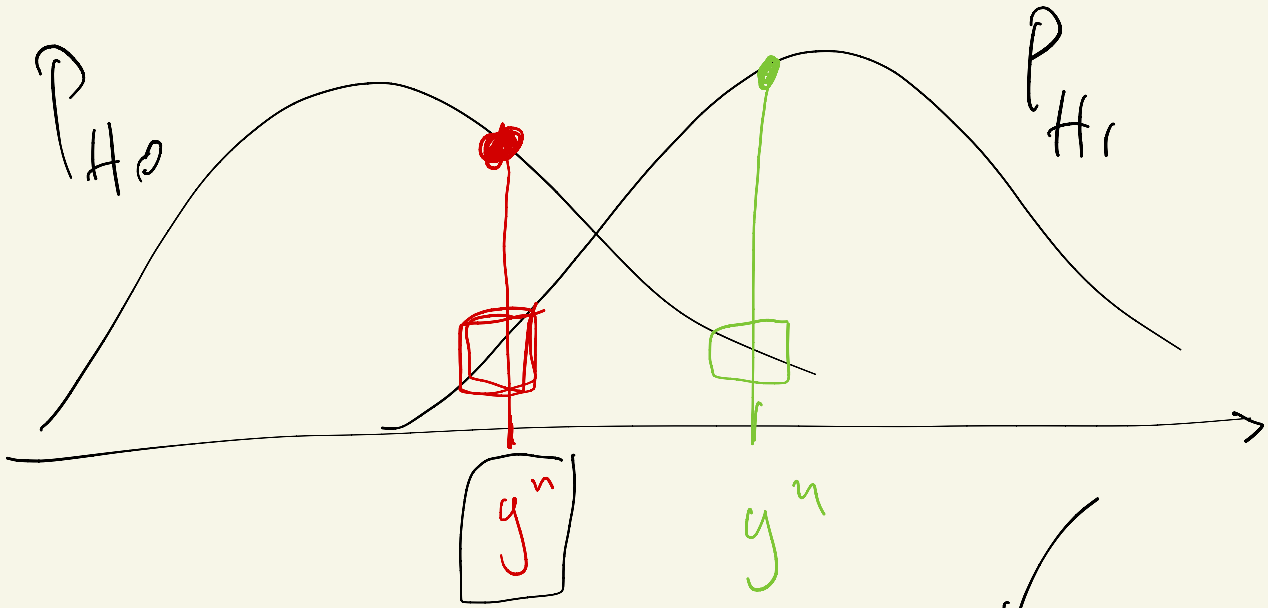
1

Let $\boxed{\mathcal{Y}^n = \text{set of values of } y^n \text{ where we declare } H_1}$



$\{0,1\}^n$

$$\boxed{P(\underbrace{Y^n \in \mathcal{Y}^n}_{\text{declare } H_1} \mid \underline{Y^n \sim H_0}) + P(\underbrace{Y^n \notin \mathcal{Y}^n}_{\text{declare } H_0} \mid \underline{Y^n \sim H_1})}$$



$$y^n = \left\{ y^n \in \{0,1\}^n : P_{H_1}(y^n) > P_{H_0}(y^n) \right\}$$

$$= \left\{ y^n \in \{0,1\}^n : \frac{1}{n}\sum_{i=1}^{n} y_i > \frac{1}{2} \right\}$$

$$P_{H_0}(y^n) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i}$$

$p \in (0, \frac{1}{2})$

$$= p^{\sum y_i}(1-p)^{n-\sum y_i}$$

$$= \left(\frac{p}{1-p}\right)^{\sum y_i}(1-p)^n$$

$$\boxed{\left(\frac{p}{1-p}\right)^{\sum y_i}} (1-p)^n < \boxed{\left(\frac{1-p}{p}\right)^{\sum y_i}} p^n$$

$$\Leftrightarrow \quad \boxed{\sum y_i < \frac{n}{2}}$$

$$\Leftrightarrow \quad \left(\frac{p}{1-p}\right)^{2\sum y_i} < \left(\frac{p}{1-p}\right)^{n}$$

$$\Leftrightarrow \quad \boxed{2\sum y_i < n}$$

$$\begin{cases} \sum y_i > \frac{n}{2} \Rightarrow H_1 \\ \sum y_i \leq \frac{n}{2} \Rightarrow H_0 \end{cases}$$

$\Leftrightarrow$ majority rule

(b) $P\left(\sum_{i=1}^{n} Y_i > \frac{n}{2} \mid Y^n \sim H_0\right)$

$+ P\left(\sum_{i=1}^{n} Y_i \leq \frac{n}{2} \mid Y^n \sim H_1\right)$

$\longrightarrow 0$

$P\left(\sum Y_i > \frac{n}{2} \mid Y^n \sim H_0\right) \to 0$

$Z_1, \dots, Z_n \overset{iid}{\sim} \mathrm{Ber}(p)$

$$P\left(\sum_{i=1}^{n} Z_i > \frac{n}{2}\right) \to 0$$

$$= P\left(\underbrace{\sum_{i=1}^{n} Z_i - np} > \frac{n}{2} - np\right)$$

$$\leq P\left(\left|\sum_{i=1}^{n} Z_i - np\right| > \frac{n}{2} - np\right)$$

$$= P\left(\left|\sum_{i=1}^{n} Z_i - np\right|^2 > \left(\frac{n}{2} - np\right)^2\right)$$

Markov

$$\leq \frac{\mathrm{Var}\left(\sum_{i=1}^{n} Z_i \cancel{-np}\right)}{\left(\frac{n}{2} - np\right)^2} \qquad \begin{matrix} n \cdot c \\ \leftarrow n^2 \cdot c \end{matrix}$$

$$\sim \frac{1}{n} \to 0$$

(c) $p = \frac{1}{2} - \frac{1}{\log(n)}$

$$\frac{n \operatorname{Var}(Z_1)}{\left(\frac{n}{2} - np\right)^2} \longrightarrow 0$$

$p(1-p)$

$\frac{\log^2}{n}$

$p = \frac{1}{2} - \frac{1}{\log(n)}$

$\boxed{\text{yes}}$